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**Application of Multi-Wavelet Analysis in Image Compression**

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**Abstract**

Wavelet analysis has been widely applied in many fields such as image processing as well as signal representation and analysis with its unique characteristics of time-frequency localization. To use multi-wavelet in the image compression is an important aspect of the application of wavelet. However, most of the multi-scale function of the existing wavelet can meet the low-pass property and how to convert one-dimensional signal as the vector input flow deserves further research. Bi-orthogonal wavelet has compact support, high-order vanishing moments and symmetry and its construction theory has attracted extensive attention and research from the people. This paper explores the applications of multi-wavelet in the image compression from the perspective of bi-orthogonal multi-wavelet and proposes the idea to use bi-orthogonal balanced multi-wavelet algorithm in the image compression. The result of the simulation experiment shows that to use this method in the image compression can obtain a higher peak signal to noise ratio and a relatively ideal compression ratio.

**Key words:** Multi-Wavelet, Image Compression, Bi-Orthogonal Balanced Multi-Wavelet.

1. **INTRODUCTION**

The theory of wavelet analysis is a mature theory in mathematics and it is a branch of applied mathematics developed in the middle and late 1980s. It not only has originality and integrity of mathematical mechanism, but also has the practicability and reality of the method as well as the convenience of the process and it overcomes the defect that Fourier transform cannot have localization features in the time and frequency domains. Wavelet analysis has excellent spatial resolution and frequency resolution and it is especially applicable for the analysis of non-stationary signals. That the natural image has such non-stationary features can be seen as linear combination of the signals of the concentration of energy space (image edges and details) and the frequency concentration (the smooth change part of the image). Therefore, wavelet analysis has been widely applied in the signal and image processing(Rajiv and Ashish, 2014; Basar and Ziya, 2015).

The concept of wavelet analysis was proposed by Morlet, a French geophysicist in 1974 for the first time and the inversion has been established through the actual experience of intuitive physics and signal processing. In 1986, a French mathematician, Meyer has successfully constructed the smooth wavelet base function with certain attenuation and worked with Mallat to build the multi-scale analysis method of wavelet base function. Soon afterwards, Mallat uses the concept of multi-resolution analysis and proposes Mallat fast decomposition and reconstruction algorithm, which has been widely applied nowadays(Mohammad and Reza et al, 2014). The single-orthogonal wavelet base function based on spline function was constructed by Jintai Cui and Jianzhong Wang in the year of 1990. In 1992, A Cohen and I Daubechhies have constructed the bi-orthogonal wavelet base of compact support. In the following development of wavelet theory, the concept of multi-wavelet emerged in order to make up for the shortcomings of single wavelet and the basic idea is to expand the multi-resolution analysis space generated from the single function in the single wavelet to that generated from multi-scale function to obtain more freedom(Ratikanta and Mani, 2013). The advantage of multi-wavelets lies in the excellent properties such as smoothness, compact support, symmetry and orthogonality and it has drawn extensive attention, however, in the practical applications, because it is quite difficult to construct multi-wavelet, there are few multi-wavelets with excellent application effects and in order to compensate for this defect, balanced multi-wavelet has been introduced and pre-filtering is not necessary for bi-orthogonal wavelet in the signal processing, suggesting the excellent application prospect(Stephan and Frank et al, 2015).

This paper firstly discusses image compression as well as the application principles of wavelet analysis in the image compression. Then, it analyzes some excellent properties of wavelet such as wavelet scaling and translation in the time domain and frequency domain as well as wavelet decomposition and reconstruction. Finally, it proposes a bi-orthogonal balanced wavelet method with lifting scheme and applies it in the image compression. The experiment simulation result shows the effectiveness of the method of this paper.

2. **IMAGE COMPRESSION**

Image compression refers to the technique to represent the pixels of the original image losslessly or lossily with fewer bits. The neighborhood pixels of the image are highly related to each other and each pixel point has
high relevance whether in the line of row or column. To replace high relevance with the low relevance of the image is to eliminate such redundant information. This is the basic idea of the theory of image compression. Is the data after image compression still in line with the original data? Is the reconstructed image distorted? The image compression can be divided into lossy compression and lossless compression.

The traditional single-wavelet has already been widely applied in the image compression and the selection of wavelet base is one of the research focuses. To settle the contradictions of large image data volume, limited communication bandwidth and limited storage space in practical applications by reducing the redundant information of the image data in order to meet the demands of most image storage and transmission requires wavelet base to have many excellent properties, including orthogonality, symmetry, smoothness and short support, which is impossible for the single wavelet. Therefore, the research of multi-wavelet theory has aroused people’s attention. The basic idea to use multi-wavelet in the image encoding is as follows: to perform multi-resolution decomposition on the image into the sub-images in different spaces and different frequencies and to conduct coefficient encoding on the sub-images(Pascal and Christian et al, 2015).

It can be seen from Figure 1 that to decompose the image into low-frequency components and the high-frequency detailed components in different directions. The coding of image transform domain is to transform the signals described in the time domain and frequency domain into those in other orthogonal vector transform domain and make each signal component described in the transform domain little relevant or irrelevant. Perform quantization and coding processing with different strategies according to the visual features of human eyes on the low- and high-frequency components. The coefficients after wavelet transform are mainly focused on the low-frequency parts and there is little energy in the horizontal, vertical and diagonal directions(Mario and Hermilo et al, 2014).

3. MULTI-WAVELET ANALYSIS

3.1. Continuous Wavelet Transform

Wavelet transform is a time-scale analysis method of signal and it has the feature of multi-resolution analysis and the ability to represent local features of the signal in the time domain and frequency domain. Besides, it is also a time-frequency localized analysis method with a fixed window and changeable shape, time window and frequency window. However, there is redundant information representation in the continuous wavelet transform and the reconstruction formula to restore the original signal by continuous wavelet transform is not unique.

If \( \psi(x) \in L^2(R) \), then wavelet is a function or a signal \( \psi(x) \) which satisfies the following conditions in the function space \( L^2(R) \).

\[
  C_\psi = \int_{R^+} \left| \hat{\psi}(\omega) \right|^2 d\omega < \infty.
\]  

(1)

In this formula, \( R^+ \) is any non-zero real number. \( \hat{\psi}(\omega) \) is the Fourier transform of \( \psi(x) \). \( \psi(x) \) is the base wavelet. It has attenuation, volatility, band-pass and energy limitability.

For the real-number pair \((a,b)\), the parameter \( a \) is a non-zero real number and the function is
\[
\psi(a,b)(x) = \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right)
\]  

(2)

We call \( \psi_{a,b}(x) \) a sub-function of the wavelet family relying on parameters \( a \) and \( b \). In the above formula, \( a \) is a scale factor used to control the shape of the image of the sub-wavelet in the shape of the wavelet image \( \psi_{a,b}(x) \) in the family and \( b \) is the translation factor to control the central position of wavelet sub-function. Fig.2 is the graph after scaling and translation of Morlet wavelet function in time domain and frequency domain (Say and Tim et al, 2013).

\[1( , ) xba b x aa
- \right) = \left( \frac{1}{2}, 1 \right)
\]  

Fig.2. Morlet wavelet in time domain and frequency domain after scaling and translation

It is can known from Figure 2 that the sequences of the sub-functions in the wavelet family obtained from the same wavelet after scaling and translation are orthogonal to each other. Moreover, the shape and size of the window of the wavelet function is changeable. For high-frequency signals, it lasts a short time and in a small scale, the time window narrows and the frequency window widens. For low-frequency signals, it lasts for a longer time and in a large scale, the time window widens and the frequency window narrows, which is good for the comprehensive description of the signal (Zhou, 2012).

Assume that \( \psi \) is the base wavelet and \( \{\psi_{a,b}\} \) is the continuous wavelet obtained from Formula (2), and then the continuous wavelet transform \( \mathcal{W}_{a,b} f \) of the function \( f(x) \in L^2(R) \) is defined as follows.

\[
\mathcal{W}_{a,b} f(x) = \int_{\mathbb{R}} f(x) \psi \left( \frac{x-b}{a} \right) dx = \langle f(x), \psi_{a,b}(x) \rangle
\]  

(3)

And its reverse transform (signal restoration or signal reconstruction) is

\[
f(x) = \frac{1}{C_a} \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}_{a,b} f(y) \psi \left( \frac{y-b}{a} \right) dydb
\]  

(4)

The introduction of factor \( \left[ \frac{1}{|a|} \right]^2 \) in Formula (3) is for the normalization and to make \( \| \psi_{a,b} \|_2 = \| \psi \|_2 \) work for all \( a \) and \( b \). Assume \( \| \psi \|_2 = 1 \), then \( \mathcal{W}_{a,b} f \leq \| f \|_2 \).

3.2. The Discretization of Continuous Wavelet Transform

Due to the redundancy in the continuous wavelet transform, certain discretization is needed to be performed on the variables \( a \) and \( b \) in the transform domain to eliminate such redundancy in order to reconstruct the signal. In practice, \( b = k \frac{2^j}{2^i}, a = \frac{1}{2^i}, j, k \in \mathbb{Z} \). At this time,

\[
\psi_{a,b}(t) = \psi_{k+\frac{1}{2^i}} \left( t - \frac{2^{i-j}}{2^i} \right) = 2^{i/2} \psi \left( 2^i t - k \right)
\]  

(5)
\( \psi_{j,k}(t) \) is usually used for short and the transform form is \( \text{WT} \left( \frac{1}{2^j}, \frac{k}{2^j} \right) = \langle f, \psi_{j,k} \rangle \). In order to reconstruct the signal \( f(t) \), \( \{ \psi_{j,k} \}_{j,k \in \mathbb{Z}} \) is required as Riesz base of \( L^2(\mathbb{R}) \).

3.3. Wavelet Decomposition and Reconstruction

The wavelet coefficients of different levels after the wavelet decomposition all include the target information in the image. Wavelet decomposition has higher frequency resolution in the low-frequency parts as well as higher time resolution and lower frequency resolution in the high-frequency parts. Proper selection of wavelet base can help focus the energy of the wavelet transform space, which is good for selecting the main components as the features. If the wavelet is decomposed downwards, the signal division will be further refined and the smoothness and stability of the detailed signal and approximation signal will be better and it is good for the signal trend analysis in a more in-depth manner. However, the computation work increases as the increase of the decomposition level and so does the error. Fig. 3 is the decomposition of wavelet transform and the storage order of sub-band coefficients of image Lena (Zou, 2013).

![Figure 3](image)

Figure 3. The decomposition algorithm of wavelet transform and the storage order of sub-band coefficients.

The decomposition process of Figure 3 is usually realized with a low-pass filter and a high-pass filter. The low-frequency coefficients include most energy of the image and they are the approximate components of the image. The high-frequency coefficients of different directions reflect the detailed information of the image. Firstly, decompose the image in the direction of row. Then decomposition it in the direction of column and obtain an approximation signal and three detailed signals in different directions. The approximation signal can be taken as the input for the decomposition of the next level. To reconstruct the original image with the wavelet coefficients is the inverse process (Siraj-ul, Imran et al., 2013).

3.4. The Construction of Bi-Orthogonal Balanced Multi-Wavelet Transform

Generally speaking, all other than the coefficient matrix two-scale matrix equation is 0. Below is the discussion of the construction of bi-orthogonal balanced multi-wavelet.

The multi-resolution analysis of space \( L^2(\mathbb{R}) \) is to construct a sub-space column \( \{ V_j \}_{j \in \mathbb{Z}} \) in this space. In order to make \( \{ V_j \}_{j \in \mathbb{Z}} \) a multi-resolution analysis of the space of \( L^2(\mathbb{R}) \), there is only one function \( \varphi(t) \in L^2(\mathbb{R}) \) to make

\[
\varphi_{j,k} = 2^{j/2} \varphi(2^j t - k), k \in \mathbb{Z}
\]  

(6)
It must be a standard orthogonal basis in $V_j$ and $\varphi(t)$ is called the scale function.

The coefficient $2^{-j/2}$ in Formula (6) is to make $L^2$ norm of $\varphi_{j,k}$. The introduction of scale function is to construct orthogonal wavelet basis.

If $\varphi(t)$ produces a multi-resolution analysis, then $\varphi \in V_0$ and because $\{\varphi_{j,k} : k \in Z\}$ is also a Riesz basis of $V_{j-1}$, there is the only $L^2$ sequence $\{h(k)\}$, which describes the dual scale relationship of the scale function $\varphi$.

$$\varphi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} h(k) \varphi(2t-k)$$  \hspace{1cm} (7)

From the above formula, it can be seen that $V_{j+1} \subset V_j$, $\forall j \in Z$, therefore

$$V_j = V_{j+1} \oplus W_{j+1}$$ \hspace{1cm} (8)

Repeat Formula (8) and obtain

$$L^2(R) = \bigoplus_{j \in \mathbb{Z}} W_j$$ \hspace{1cm} (9)

Just as $\varphi(t)$ produces $V_0$, there exists a function $\psi(t)$ which produces a closed sub-space $W_0$ and which has a dual scale equation similar to Formula (7).

$$\psi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} g(k) \varphi(2t-k)$$ \hspace{1cm} (10)

Formula (10) is called the dual scale equation of the wavelet function.

If $(h, g)$ is a certain known bi-orthogonal multi-filter bank with first-order approximation, $(h^{old}, g^{old})$ is the corresponding dual multi-filter bank. If the lifting of operator $k$ meets the following

$$\begin{bmatrix} h^T \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \\ g^T \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$ \hspace{1cm} (11)

then the multi-filter bank $(h^{new}, g^{new})$ obtained through lifting and dual lifting is bi-orthogonal and balanced.

$$\begin{cases} h^{new} = h + g \cdot k' \\ g^{new} = g + g \cdot k' \end{cases}$$ \hspace{1cm} (12)

4. EXPERIMENT SIMULATION AND ANALYSIS

In most cases, people will use some objective criteria to evaluate the quality of the restored image. The mean square error (MSE) between the data of the original image and the reconstructed image is defined as follows.

$$MSE = \frac{1}{N_x N_y} \sum_{j=1}^{N} \sum_{i=1}^{N} [f(i,j) - f'(i,j)]^2$$ \hspace{1cm} (13)

The peak signal to noise ratio (PSNR) is defined as below.
\[
PSNR = 20 \log \left( \frac{2^n - 1}{\sqrt{MSE}} \right) dB
\] \hspace{1cm} (14)

In this formula, \( f(i, j) \) is the value of the pixel point \((i, j)\) of the original image, \( f'(i, j) \) is the value of the pixel point \((i, j)\) of the reconstructed image, \( N_x \) and \( N_y \) are the numbers of rows and columns of the image respectively and \( n \) is the number of digits.

This paper uses the following experiment environment: the frequency of CPU is 3.3GHz, the computer memory is 4G and the operating system is Win7 and Matlab2012a. Fig.4 and Fig.5 is the comparison result of test standard images at the compression ratio of 8:1 and after the bi-orthogonal balanced multi-wavelet (BOBMW) and the common multi-wavelet reconstruction.

![Figure 4](image1.png)

**Figure 4.** Comparison of test image 1 at the compression ratio of 8:1

![Figure 5](image2.png)

**Figure 5.** Comparison of test image 2 at the compression ratio of 8:1

It can be seen from the above that after using bi-orthogonal balanced multi-wavelet algorithm, the image quality has been improved with the same compression ratio. Table 1 shows the signal to noise ratio of the restored image by using two algorithms in different compression ratios.

<table>
<thead>
<tr>
<th>Compression ratio</th>
<th>Pixel bit rate</th>
<th>PSNR</th>
<th>Multi-wavelet</th>
<th>BOBMW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test image 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8:1</td>
<td>1</td>
<td>40.41</td>
<td>40.95</td>
<td></td>
</tr>
<tr>
<td>16:1</td>
<td>0.5</td>
<td>37.37</td>
<td>37.89</td>
<td></td>
</tr>
<tr>
<td>32:1</td>
<td>0.25</td>
<td>33.76</td>
<td>34.48</td>
<td></td>
</tr>
<tr>
<td>64:1</td>
<td>0.125</td>
<td>31.62</td>
<td>32.70</td>
<td></td>
</tr>
<tr>
<td>Test image 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8:1</td>
<td>1</td>
<td>38.88</td>
<td>39.46</td>
<td></td>
</tr>
<tr>
<td>16:1</td>
<td>0.5</td>
<td>34.63</td>
<td>35.17</td>
<td></td>
</tr>
<tr>
<td>32:1</td>
<td>0.25</td>
<td>31.56</td>
<td>32.04</td>
<td></td>
</tr>
<tr>
<td>64:1</td>
<td>0.125</td>
<td>29.83</td>
<td>30.32</td>
<td></td>
</tr>
</tbody>
</table>

The experiment result demonstrates that compared with the common multi-wavelet algorithm, the PSNR of the reconstructed image which uses BOBMW algorithm has been improved to a certain extent in the same compression ratio. Besides, with the improvements of the compression ratio, PSNR increases at an increasing
trend. To use BOBMW algorithm can effectively improve the efficiency of the image encoding, namely restored image can also be obtained at a lower encoding rate.

To analyze the above results, it can be seen whether from the quality of the reconstructed image subjectively or from the objective parameters that BOBMW has improved compared to multi-wavelet algorithm. Therefore, BOBMW has more advantages and these advantages are increasingly obvious in higher level, namely at a higher compression rate.

5. CONCLUSIONS

Multi-wavelet transform is a strong tool in analyzing and processing non-stationary signals. It is performed based on the wavelet base formed by localization function and with many special performances and advantages, it is a more reasonable multi-resolution analysis method of time-frequency representation and sub-band. On the basis of classical multi-wavelet algorithm, this paper applies bi-orthogonal balanced multi-wavelet algorithm into the image compression and enhances the effect of the image compression and reconstruction. Besides, this paper proves the validity of the construction of bi-orthogonal balanced multi-wavelet algorithm and the superiority of its properties from the theory and the experiment.

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