Edge Chromatic Number on Some Equal Degree Graph

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Abstract
Let $G(V, E)$ be a simple graph and $k$ is a positive integer, if it exists a mapping $f$, and satisfied with $f(e_1) \neq f(e_2)$, for two adjacent edges $e_1, e_2 \in E(G)$, then $f$ is called the $k$-proper-edge coloring of $G(K - \mathrm{PEC}$ for short). The minimal number of colors required for a proper edge coloring of $G$ is denoted by $\chi'(G)$ and is called the proper edge chromatic number. It exists a $k$-proper edge coloring of simple graph $G$, for any two adjacent vertices $u$ and $v$ in $G$, the set of colors assigned to the edges incident to $u$ differs from the set of colors incident to $v$, then $f$ is called $k$-adjacent-vertex-distinguishing proper edge coloring, is abbreviated $k$-AVDEC, also called a adjacentstrong edge coloring. The minimal number of colors required for a adjacent-vertex-distinguishing edge coloring of $G$ is denoted by $\chi'_{av}(G)$ and is called adjacent-vertex-distinguishing edge chromatic number. The new class graphs of equal degree graph are introduced, and this class graphs adjacent-vertex-distinguishing edge chromatic numbers of path, cycle, fan, complete graph, wheel, star are presented in this paper.

Key words: Graph Wheel, Join-graph, Adjacent Vertex-distinguishing, Edge Chromatic Number.

1. INTRODUCTION

The coloring problem of graphs is an extremely difficult problem, widely applied in practice. Some conditional coloring problems are introduced by Harary (Harary, 1985). Some network problems can be converted to the edge coloring (Burris and Schelp, 1997; Bazgan C Harkat-Benhamdine and Li 1999; Balister, 2003) and adjacent strong edge coloring.

2. PRELIMINARIES

With Definition 1 (Bondy J A and Marty U S R, 1976) For a graph $G(V, E)$, if a proper coloring $f$ is satisfied with $C(u) \neq C(v)$, for $u, v \in V(G)(u \neq v)$ and $u, v \in E(G)$, then $f$ is called $k$-adjacent-vertex-distinguishing edge coloring of $G$, is abbreviated $k$-AVDEC, and

$$\chi'_{av}(G) = \min \{k/k - \mathrm{AVDEC} \text{ of } G\}$$

is called the adjacent-vertex-distinguishing edge chromatic number of $G$ [6], where

$$C(u) = \{f(uv)|uv \in E(G)\}.$$

Conjecture (Jerner, 1996) Let $G$ be a connected graph with $|G| \geq 3$, and $G \neq C_5(5$-cycle), then

$$\chi'_{av}(G) \leq \Delta(G) + 2$$

where $\Delta(G)$ is the maximum degree of graph $G$.

Definition 2 (Balister and Schelp, 2002) Let $G(V, E)$ be a simple graph, $M(G)$ is called the equal degree graph of $G$, where

$$V(M(G)) = E(G) \cup V' \cup \{v\}$$

$$E(M(G)) = E(G) \cup \{uv'|u \in V(G), v' \in V, d(u) = d(v)\} \cup \{wv'|v' \in V'\}$$

Whereas $V' = \{v' | v \in V(G)\}, \{w\} \cap (V(G) \cup V') = \phi$. 


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3. MAIN RESULT AND CONCLUSIONS

Lemma1 For \( n \geq 2 \), then
\[
\Delta(M(P_n)) = \begin{cases} 
3, n = 2, 3 \\
n, n \geq 4 
\end{cases}
\]

Lemma2 For \( n \geq 3 \), then
\[
\Delta(M(C_n)) = n + 2
\]

Lemma3 For \( n \geq 3 \), then
\[
\Delta(M(F_n)) = \begin{cases} 
5, n = 2, 3 \\
n + 1, n \geq 4 
\end{cases}
\]

Lemma4 For \( n \geq 2 \), then
\[
\Delta(M(K_n)) = \begin{cases} 
3, n = 2 \\
5, n = 3 \\
2n - 1, n \geq 4 
\end{cases}
\]

Lemma5 For \( n \geq 3 \), then
\[
\Delta(M(W_n)) = \begin{cases} 
7, n = 3 \\
n + 3, n \geq 4 
\end{cases}
\]

Lemma6 For \( n \geq 3 \), then
\[
\Delta(M(S_n)) = \begin{cases} 
3, n = 1, 2 \\
n + 1, n \geq 3 
\end{cases}
\]

Lemma7 Let \( G \) be a connected simple graph with \( |V(G)| \geq 3 \), if \( uv \in E(G) \) and \( d(u) = d(v) = \Delta(G) \), then
\[
\chi'_d(G) \geq \Delta(G) + 1
\]

Theorem 1 For \( n \geq 2 \), then
\[
\chi'_d(M(P_n)) = \begin{cases} 
5, n = 2, 3 \\
n + 1, n \geq 4 
\end{cases}
\]

Proof Let \( V(P_n) = \{u_1, u_2, \ldots, u_n\}, V' = \{v_1, v_2, \ldots, v_n\} \), there are two cases to be discussed as follow:

Case1 When \( n = 2 \), \( \Delta(M(P_2)) = 3 \) from lemma 1. However, \( u_1, u_2, v_1, v_2 \) are vertices which are maximum degree of \( M(P_2) \) and all adjacent, \( \chi'_d(M(P_2)) \geq \Delta(M(P_2)) = 4 \) from lemma 7. If \( M(P_2) \) exists 4-AVDEC, because four vertices of maximum degree are adjacent each other, and \( \binom{4}{3} = 4 \). The sets are \( \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \). Let every set correspond a vertex, we can obtain the vertices of coloring same color are odd, but the vertices of coloring same color are even by graph theory, so \( M(P_2) \) don’t exists 4-AVDEC. In order to prove the conclusion be true, we only need construct a map \( f \) from \( E(M(P_2)) \) to \( \{1, 2, 3, 4, 5\} \):
\[
\begin{align*}
& f(u_1v_1) = f(u_2v_2) = 1; \quad f(u_2v_1) = 2; \quad f(u_1v_2) = 3; \quad f(u_2u_3) = 4; \quad f(wv_1) = 3; \quad f(wv_2) = 5; \\
& \text{then we have} \\
& \overline{C}(u_1) = \{3, 5\}; \quad \overline{C}(u_2) = \{2, 5\}; \quad \overline{C}(v_1) = \{4, 5\}; \quad \overline{C}(v_2) = \{2, 4\}
\end{align*}
\]

Obviously, the \( f \) is 5-AVDEC of \( M(P_2) \), the conclusion is true.

When \( n = 3, \Delta(M(P_3)) = 3 \) from lemma 1. But \( M(P_2) \subset M(P_3) \), \( M(P_3) \geq 5 \) from \( n=2 \). It is obviously to prove exists 5-AVDEC of \( M(P_3) \), we only need to construct a map \( f \) from \( E(M(P_3)) \) to \( \{1, 2, 3, 4, 5\} \):
\[
\begin{align*}
& f(u_1u_2) = f(u_3v_1) = 1; \quad f(u_2u_3) = f(u_1v_1) = 2; \quad f(u_2v_1) = f(u_2u_3) = 3; \\
& f(wv_1) = f(wv_2) = 5 \\
& \text{then we have} \\
& \overline{C}(u_1) = \{4, 5\}; \quad \overline{C}(u_2) = \{3, 5\}; \quad \overline{C}(u_3) = \{3, 4\}; \quad \overline{C}(v_1) = \{1, 3\}; \quad \overline{C}(v_2) = \{1, 2, 5\}; \quad \overline{C}(v_3) = \{2, 4\}
\end{align*}
\]
Obviously, the $f$ is $5-$AVDEC of $M(P_5)$, the conclusion is true.

Case 2 When $n \geq 4$, $\Delta(M(P_n)) = n$ from lemma 1. However the vertices of $u_2, u_3, \cdots, u_{n-1}$ are vertices which are maximum degree and exist some are adjacent, $\chi''(M(P_n)) \geq n+1$ from lemma 7. It is obviously to prove exists $(n+1)-$AVDEC of $M(P_n)$, we only need to construct a map $f$ from $E(M(P_n))$ to $\{1, 2, \cdots, n, 0\}$:

$$
\begin{align*}
  f(u_i u_i) &= i - 1, i = 1, 2, \cdots, n-1; \\
  f(u_i v_i) &= 1; \\
  f(u_j u_i) &= i + j \mod n, i = 2, 3, \cdots, n-1, j = 2, 3, \cdots, n-1; \\
  f(u_i v_j) &= n; \\
  f(w_j v_i) &= i-1, i = 1, 2, \cdots, n
\end{align*}
$$

so we have

$$
\begin{align*}
  \overline{C}(u_i) &= \{0, 1, 2\}; \\
  \overline{C}(u_j) &= \{n\}; \\
  \overline{C}(u_i) &= \{i-3\}, i = 3, 4, \cdots, n-1; \\
  \overline{C}(u_s) &= \{0, n-2, n-1\}; \\
  \overline{C}(v_i) &= \{0, 1, n\}; \\
  \overline{C}(v_j) &= \{i-1, \cdots, n+i-3 \mod n+1\}, i = 2, 3, \cdots, n-1; \\
  \overline{C}(v_s) &= \{0, 2, n-1\}.
\end{align*}
$$

Obviously, the $f$ is $(n+1)-$AVDEC of $M(P_n)$, the conclusion is true.

From all of above two cases, the conclusion is true.

**Theorem 2** For $n \geq 3$, then

$$
\chi''(M(C_n)) = n+3
$$

**Proof** Let $V(C_n) = \{u_1, u_2, \cdots, u_n\}, V' = \{v_1, v_2, \cdots, v_n\}$, $u_1, u_2, \cdots, u_n$ are vertices of maximum degree and exists some adjacent, $(M(C_n)) \geq n+3$ from lemma 2 and 7. It is obviously to prove exists $(n+3)-$AVDEC of $M(C_n)$, we only need to construct a map $f$ from $E(M(C_n))$ to $\{1, 2, \cdots, n+2, 0\}$:

$$
\begin{align*}
  f(u_i u_i) &= i-1, i = 1, 2, \cdots, n-1; \\
  f(u_i v_i) &= 0; \\
  f(u_i v_j) &= i+j \mod n+3, i = 2, 3, \cdots, n-1, j = 1, 2, \cdots, n
\end{align*}
$$

so we have

$$
\begin{align*}
  \overline{C}(u_i) &= \{n+2\}; \\
  \overline{C}(u_j) &= \{i-1\}; \\
  \overline{C}(u_i) &= \{i, n\}; \\
  \overline{C}(u_s) &= \{n\}
\end{align*}
$$

Obviously, the $f$ is $(n+3)-$AVDEC of $M(C_n)$, the conclusion is true.

**Theorem 3** For $n \geq 3$, then

$$
\chi(M(F_n)) = \begin{cases}
  6, n = 2, 3 \\
  n+2, n \geq 4
\end{cases}
$$

**Proof** Let $V(F_n) = \{u_0, u_1, u_2, \cdots, u_n\}, V' = \{v_0, v_1, v_2, \cdots, v_n\}$, there are three cases to be discussed as follow:

Case 1 When $n = 2$, $M(F_2) = M(C_3)$, so we can obtain $\chi'(M(F_2)) = \chi'(M(C_3)) = 6$, then the conclusion is true.

Case 2 When $n = 3$, $\Delta(M(F_3)) = 5$ from lemma 3, and $M(F_1)$ exists two vertices of maximum degree are adjacent, so $\chi''(M(F_3)) \geq 6$ by lemma 7. It is obviously to prove exists 6-AVDEC of $M(F_3)$, we only need to construct a map $f$ from $E(M(F_3))$ to $\{1, 2, 3, 4, 5, 0\}$:

$$
\begin{align*}
  f(u_0 u_1) &= i-1, i = 1, 2, 3; \\
  f(u_1 u_2) &= i+1, i = 1, 2; \\
  f(u_0 v_0) &= 3; \\
  f(u_0 v_3) &= 4; \\
  f(u_0 v_4) &= 5; \\
  f(u_1 v_0) &= 4; \\
  f(u_1 v_5) &= 5; \\
  f(u_0 v_5) &= 4; \\
  f(u_1 v_3) &= 5; \\
  f(w_0 v_0) &= i-1, i = 0, 1, 2, 3
\end{align*}
$$

then we have

$$
\overline{C}(u_0) = \{5\}, \overline{C}(u_2) = \{0\}
$$

Obviously, the $f$ is 6-AVDEC of $M(F_3)$, the conclusion is true.

Case 3 When $n \geq 4$, $\Delta(M(F_n)) = n + 1$ from lemma 3, $M(F_n)$ exists some vertices of maximum degree are adjacent. $\chi'(M(F_n)) \geq \Delta(M(F_n)) + 1 = n + 2$ by lemma 3, 7. It is obviously to prove exists $(n+2)-$AVDEC of $M(F_n)$, we only need to construct a map $f$ from $E(M(F_n))$ to $\{1, 2, 3, \cdots, n+1, 0\}$:

$$
\begin{align*}
  f(u_0 v_0) &= i-1, i = 1, 2, \cdots, n; \\
  f(u_1 v_0) &= i+1, i = 1, 2, \cdots, n; \\
  f(u_0 v_n) &= n; \\
  f(u_1 v_3) &= 4; \\
  f(u_0 v_3) &= 4; \\
  f(u_0 v_5) &= 5; \\
  f(u_1 v_5) &= 5; \\
  f(w_0 v_0) &= i-1, i = 2, 3, \cdots, n-1
\end{align*}
$$
\[ f(u_i, v_i) = 1; \quad f(u_i, v_n) = 2; \quad f(w_i) = i + 1, i = 0, 1, 2, \ldots; \]

so we have

\[ \overline{C}(u_i) = [n + 2], \overline{C}(u_n) = [i - 2], i = 2, 3, \ldots, n - 1, \overline{C}(w) = [0] \]

Obviously, the \( f \) is \( n + 2 \)-AVDEC of \( M(F_n) \), the conclusion is true.

From all of above, the conclusion is true.

**Theorem 4** For \( n \geq 2 \), then

\[ \chi'(M(K_n)) = \begin{cases} 5, & n = 2; \\ 6, & n = 3; \\ \Delta(M(K_n)) + 1, & n \geq 4 \end{cases} \]

**Proof** Let \( V(K_n) = \{u_1, u_2, \ldots, u_n\}, V' = \{v_1, v_2, \ldots, v_n\} \), there are three cases to be discussed as follow:

**Case 1** When \( n = 2 \), \( M(K_2) = M(P_2) \), so we can obtain \( \chi'(M(K_2)) = \chi'(M(P_2)) = 5 \), the conclusion is true.

**Case 2** When \( n = 3 \), \( M(K_3) = M(C_3) \), so we can obtain \( \chi'(M(K_3)) = \chi'(M(C_3)) = 6 \), the conclusion is true.

**Case 3** When \( n \geq 4 \), \( \Delta(M(K_n)) = 2n - 1 \) from lemma 4, \( M(K_n) \) exists some vertices of maximum degree are adjacent, \( \chi'(M(K_n)) \geq \Delta(M(K_n)) + 1 = 2n \) by lemma 4.7. It is obviously to prove exists \( 2n \)-AVDEC of \( M(K_n) \), we only need to construct a map \( f \) from \( E(M(K_n)) \) to \( \{1, 2, \ldots, 2n - 1, 0\} \):

\[ f(u_i, u_j) = i + j - 3, i = 1, 2, \ldots, n - 1; j = i + 1, i + 2, \ldots, n; \]

\[ f(u_i, v_j) = i + j + n - 3, i = 1, 2, \ldots, n; j = 1, 2, \ldots, n - 1; \]

\[ f(v_i, u_j) = 2n - 1, f(v_i, v_j) = 2(i - 1), i = 2, 3, \ldots, n; \]

\[ f(w_i) = i, i = 1, 2, \ldots, n - 1; \]

\[ f(w_n) = n, n = 0, n = 1 \mod 2; \quad f(w_n) = n + 1, n = 1 \mod 2. \]

then we have

\[ \overline{C}(u_i) = [2n + i - 3], i = 1, 2, \ldots, n \]

Obviously, the \( f \) is \( 2n \)-AVDEC of \( M(K_n) \), the conclusion is true.

From all of above, the conclusion is true.

**Theorem 5** For \( n \geq 3 \), then

\[ \chi'(M(W_n)) = \begin{cases} 8, & n = 3; \\ n + 4, & n \geq 4. \end{cases} \]

**Proof** Let \( V(W_n) = \{u_1, u_2, \ldots, u_n\}, V' = \{v_0, v_1, v_2, \ldots, v_n\} \), there are two cases to be discussed as follow:

**Case 1** When \( n = 3 \), \( M(W_3) = M(K_3) \), so we can obtain \( \chi'(M(W_3)) = \chi'(M(K_3)) = 8 \), the conclusion is true.

**Case 2** When \( n \geq 4 \), \( \Delta(M(W_n)) = n + 3 \) from the lemma 5, \( M(W_n) \) exists some vertices of maximum degree are adjacent, \( \chi'(M(W_n)) \geq \Delta(M(W_n)) + 1 = n + 4 \) by lemma 5.7. It is obviously to prove exists \( n + 4 \)-AVDEC of \( M(W_n) \), we only need to construct a map \( f \) from \( E(M(W_n)) \) to \( \{1, 2, \ldots, n + 3, 0\} \):

\[ f(u_1, u_i) = i - 1, i = 2, 3, \ldots, n; f(u_0, u_n) = n; \]

\[ f(u_i, v_j) = i + j - 1, i = 1, 2, \ldots, n - 1; j = 1, 2, \ldots, n; \]

\[ f(u_i, v_n) = n; f(u_i, v_j) = n + j, n = 2, 3, \ldots, n - 1; f(u_0, v_n) = n - 3 \]

\[ f(v_i, u_j) = n + i + 1, i = 1, 2, \ldots, n - 1; \]

\[ f(u_i, u_n) = n + 1; \]

\[ f(w_i) = i, i = 1, 2, \ldots, n - 1; f(w_j) = n + 2 + i, n = 1, \ldots, n \]

then we have

\[ \overline{C}(u_i) = [n + i + 2], i = 1, 2, \ldots, n \]

Obviously, the \( f \) is \( n + 3 \)-AVDEC of \( M(W_n) \), the conclusion is true.

From all of above, the conclusion is true.

**Theorem 6** For \( n \geq 1 \), then

\[ \chi'(M(S_n)) = \begin{cases} 5, & n = 1, 2; \\ n + 3, & n \geq 3. \end{cases} \]
Proof Let $V(S_n) = \{u_0, u_1, u_2, \ldots, u_n\}, V' = \{v_0, v_1, v_2, \ldots, v_n\}$, there are three cases to be discussed as follow:

Case1 When $n = 1$, $M(S_1) = M(P_2)$, so we can obtain $\chi'(M(S_1)) = \chi'(M(P_2)) = 5$, the conclusion is true.

Case2 When $n = 2$, $M(S_2) = M(P_3)$, so we can obtain $\chi'(M(S_2)) = \chi'(M(P_3)) = 6$, the conclusion is true.

Case3 When $n \geq 3$, $\Delta(M(S_n)) = n + 1$, by lemma 6, $M(S_n)$ exists some vertices of maximum degree are adjacent, $\chi'(M(S_n)) \geq \Delta(M(S_n)) + 1 = n + 2$ by lemma 6.7. If exists -AVDEC, then

$$\bar{C}(u_i) = \bar{C}(u_j) = \bar{C}(u_k) = 1, i, j, k = 1, 2, \ldots, n$$

$\bar{C}(u_i) \not\in \bar{C}(u_j) \cup \bar{C}(u_k) \cup \cdots \cup \bar{C}(v_s)$;

$\bar{C}(u_j) \not\in \bar{C}(v_i) \cup \bar{C}(v_k) \cup \cdots \cup \bar{C}(v_s)$;

$\bar{C}(u_k) \not\in \bar{C}(v_i) \cup \bar{C}(v_j) \cup \cdots \cup \bar{C}(v_s)$;

$\bar{C}(v_i) \not\in \bar{C}(v_j) \cup \bar{C}(v_k) \cup \cdots \cup \bar{C}(v_s)$;

(2)

$$\bar{C}(u_i) \not\in \bar{C}(u_j) \cup \bar{C}(u_k) \cup \cdots \cup \bar{C}(u_s) \cup \bar{C}(v_i) \cup \bar{C}(v_j) \cup \cdots \cup \bar{C}(v_s)$$

from (1) and (2), we can obtain

$$|\bar{C}(u_i) + \bar{C}(u_j) + \cdots + \bar{C}(u_s) + \bar{C}(v_i) + \cdots + \bar{C}(v_s)| \geq n + 2.$$

It is obvious that $M(S_n)$ don’t exist $(n + 2)$-AVDEC. In order to prove the conclusion be true, we only need to construct a map $f$ from $E(M(S_n))$ to $\{1, 2, \ldots, n + 2, 0\}$:

$$f(u, u) = i - 1, i = 1, 2, \ldots, n; f(u, v) = n;$$

$$f(u, v_j) = i + j \bmod n + 3, i = 1, 2, \ldots, n; j = 1, 2, \ldots, n;$$

$$f(v_i, v_j) = i - 1, i = 0, 1, \ldots, n;$$

then we have

$$\bar{C}(u_i) = \{n + 1, n + 2\}; \bar{C}(u_j) = \{1, n + 2\}; \bar{C}(v_i) = \{i - 2, i\}, i = 2, 3, \ldots, n;$$

$$\bar{C}(v_j) = \{0, n + 2\}; \bar{C}(v_k) = \{i - 2, i - 1\}, i = 2, 3, \ldots, n.$$

Obviously, the $f$ is $(n + 3)$ -AVDEC of $M(S_n)$, the conclusion is true.

From all of above, the conclusion is true.

From the conclusions of Theorem 1, 2, 3, 4, 5, 6, the conjecture is true for equal degree graph.

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