On the HK and the CMG Rate Regions

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Abstract
The celebrated Han and Kobayashi (HK) rate region and its equivalent description Chong, Motani, Garg and Gamal (CMG) rate region are the best achievable rate regions known to date for the two-user interference channel. They were obtained by simultaneous superposition coding or sequential superposition coding together with simultaneous decoding, respectively. In this paper, we first show that the HK and the CMG rate regions can also be obtained via successive cancelation decoding in conjunction with the corresponding superposition coding. Then, we show the equivalence between the two rate regions by analyzing the decoding orders of the messages.

Key words: Interference Channel, Capacity Region, Achievable Rate Region, Superposition Coding.

1. INTRODUCTION

When some senders communicate only with their corresponding receivers via a common channel, each user pair will be subject to interference from other user pairs. This scenario is modeled as the interference channel (IC) that was first introduced by Shannon in 1961 (Shannon, 1961). In general, the capacity region of the 2-user IC is not known, except for the following several special cases. For the 2-user discrete memoryless IC (DM-IC), the capacity region is known if the interference is strong (Sato, 1978; Costa and Gamal, 1987; Carleial, 1975), if the channel outputs are deterministic functions of the channel inputs (Gamal and Costa, 1982), or if the channel is degraded (Liu and Ulukus, 2008; Benzel, 1979). For the 2-user Gaussian IC (G-IC), the capacity regions of very strong and strong G-ICs (Carleial, 1975; Han and Kobayashi, 1981; Sato, 1981) and the sum capacities of noisy interference (Shang, Kramer and Chen, 2009; Motahari and Khandani, 2009; Annapureddy and Veeravalli, 2009), mixed interference (Motahari and Khandani, 2009; Tuninetti and Weng, 2008) and one-sided (Sason, 2004) G-ICs are known. Surprisingly, all of these results can be obtained by using simple point-to-point code in conjunction with successive cancelation decoding or treating the interference as noise.

In achieving the capacity region of the IC, superposition coding (Cover, 1975) played a significant role. Carleial first introduced a restricted version of the general superposition coding named sequential superposition coding into the context of the G-IC (Carleial, 1978), where the messages of each user are split into the common messages and the private messages based on rate-splitting technique. He exploited a successive cancellation decoder, in which each receiver decodes the intended common messages or the undesired common messages before decoding the intended private messages. The region established by Carleial is the convex hull of the union of the rate regions generated by four decoding strategies and is better than those before. In 1981, Han and Kobayashi (HK) exploited another form of superposition coding (referred to as simultaneous superposition coding) together with simultaneous decoding and obtained the best achievable rate region known to date, i.e., the celebrated HK rate region (Han and Kobayashi, 1981). In their paper, the common messages and the private messages are generated according to the coded time-sharing random variables and superposed simultaneously in a deterministic way, and each receiver simultaneously decodes the intended common messages, the undesired common messages and the intended private messages. An equivalent rate region named CMG region is proposed, which is obtained by using sequential superposition coding in conjunction with coded time-sharing and simultaneous decoding (Chong, Motani, Garg and Gamal, 2008; Kramer, 2006). However, both the HK and the CMG rate regions are complex and intangible. Many researchers have used the successive cancelation decoder to achieve the HK inner bound. A greedy decoding order algorithm was developed for a K-user memoryless IC, where each receiver decoded a subset of all transmitters sequentially before decoding the message of the designated transmitter ( Maddah–Ali, Mahdavi and Khandani, 2007). Meanwhile, two algorithms determining the decoding order for rate-splitting with partial interference decoded were proposed (Maddah–Ali, Mahdavi and Khandani, 2007). A ratesplitting based scheme with successive cancelation decoder was proposed, in which an iterative multiple water-level water-filling algorithm was introduced to optimize the power allocation (Jing, Bai and Ma, 2010). In 2012, we proposed an optimal power allocation scheme
with the proposed decoding order achieve the sum-rate within two bits per channel use of the sum capacity for two-user symmetric Gaussian interference channel with common messages (Zhang, Li, Mao and Bai, 2012).

In this paper, we will show that the HK and the CMG rate regions can be achieved by successive cancellation decoding. The equivalence between the two rate regions is provided by analyzing the decoding orders of the messages. Figure 1 is an overview on our work. To achieve the HK and the CMG regions, we explored the same encoding methodology of successive cancellation decoding instead of simultaneous decoding. In fact, since both superposition coding schemes need to decode the common and private messages intended and the partial interference messages, the 2-user IC can be viewed as a coupling channel consisting of two virtual three-user multiple access channels.

![Figure 1. Overview on the work in this paper](image)

2. PRELIMINARIES

In the following discussions, we use $X$ to denote a random variable with finite alphabet $X$ and probability mass function (pmf) $p_x(x) = \Pr(X = x)$, $x \in X$. $X^n = (X_1, X_2, ..., X_n)$ denotes a $n$-sequence.

2.1. Jointly Letter-typical Sequences

Let $N(x, y|x^n, y^n)$ be the number of times that the pair $(x, y)$ occurs in a sequence of pairs $(x_i, y_i)$, $i = 1, 2, ..., n$. Let $(X, Y) \in p_{X,Y}(x, y)$. The set $T_o^{(n)}(X, Y)$ (or $T_o^{(s)}$ in short) of jointly $\delta$-typical $n$-sequences $(x^n, y^n)$ is defined as below (Gamal and Kim, 2011)

$$T_o^{(s)} = \{(x^n, y^n) \in X^n \times Y^n : \frac{1}{n} N(x, y|x^n, y^n) - p_{X,Y}(x,y) \leq \delta \cdot p_{X,Y}(x,y) \text{ for all } x \in X, y \in Y\}.$$ (1)

2.2. Packing Lemma

The packing lemma was used to bound the probability of decoding error events (Gamal and Kim, 2011). We rewrite it as follows. Let $(U, X, Y) \in p_{U,X,Y}(u, x, y)$. Moreover, let $(\tilde{U}^n, \tilde{Y}^n) \in p_{\tilde{U},\tilde{Y}}(\tilde{u}^n, \tilde{y}^n)$ be a pair of arbitrarily distributed random sequences. Assume that $X^n(m)$, $m \in \{1, 2, ..., 2^m\}$, is pairwise conditionally independent of $\tilde{Y}^n$ given $\tilde{U}^n$, but is arbitrarily dependent on other $X^n(m)$ sequences. Then, there exists $\delta'(\delta) \to 0$ as $\delta \to 0$ such that

$$\Pr\left\{\left|\tilde{U}^n, X^n(m), \tilde{Y}^n\right| \in T_o^{(s)} \text{ for some } m \in M\right\} \to 0 \text{ as } n \to \infty \text{ if } R \leq I(X; Y|U) - \delta'(\delta).$$

3. CHANNEL MODEL

We consider a 2-user discrete memoryless interference channel (DM-IC) as Figure 2, which consists of two input $X_1$ and $X_2$, and two output alphabets $Y_1$ and $Y_2$, and a set of conditional probability mass functions (pmfs) $p(y_1, y_2|x_1, x_2)$ on $Y_1 \times Y_2$. Transmitter $j$, $j = 1, 2$ wants to send an independent message $M_j$ uniformly from $W_j$ to its intended receiver $j$. A $(2^{m_H}, 2^{m_R}, n)$ code for the DM-IC consists of two message sets
\( W_1 = \{1,2,\ldots,2^{n_1}\} \) and \( W_2 = \{1,2,\ldots,2^{n_2}\} \), two encoding functions \( f_1: W_1 \rightarrow X_1^n \) and \( f_2: W_2 \rightarrow X_2^n \), and two decoding functions \( g_1: Y_1^n \rightarrow W_1 \) and \( g_2: Y_2^n \rightarrow W_2 \). The average probability of error is defined as \( P_e^n = \Pr \{ (\hat{M}_1, \hat{M}_2) \neq (M_1, M_2) \} \). A rate pair \((R_1, R_2)\) is said to be achievable for the DM-IC if there exists a sequence of \((2^{n_1}, 2^{n_2}), n\) codes with \( P_e^n \rightarrow 0\). The capacity region of a DM-IC is the closure of the set of achievable rate pairs \((R_1, R_2)\).

\[ \text{Figure 2. The 2-user discrete memoryless IC} \]

### 4. THE MAIN RESULTS

**4.1. To Achieve the HK Rate Region**

In 1981, Han and Kobayashi introduced another form of superposition coding, i.e., simultaneous superposition coding. In the coding strategy, five auxiliary random variables \( Q, W_1, W_2, U_1 \) and \( U_2 \) defined on arbitrary finite sets \( Q, W_1, W_2, U_1 \) and \( U_2 \), respectively. Sender 1 splits the message \( V_1 \) into \((V_{11}, V_{12})\), where \( V_{11} = \{1,2,\ldots,2^{n_1}\} \) and \( V_{12} = \{1,2,\ldots,2^{n_2}\} \). Similarly, sender 2 splits the message \( V_2 \) into \((V_{21}, V_{22})\), where \( V_{21} = \{1,2,\ldots,2^{n_1}\} \) and \( V_{22} = \{1,2,\ldots,2^{n_2}\} \). The auxiliary random variable \( W_1 \) is used to carry the message \( V_{12} \), while the auxiliary random variable \( U_1 \) is used to carry the message \( V_{11} \). Similarly, the auxiliary random variable \( W_2 \) is used to carry the message \( V_{21} \), while the auxiliary random variable \( U_2 \) is used to carry the message \( V_{22} \). Clearly, both \( W_1 \) and \( W_2 \) are used to carry the common messages, while both \( U_1 \) and \( U_2 \) are used to carry the private messages. The encoding functions \( f_1 \) and \( f_2 \) are given by

\[ f_1: V_1 \rightarrow X_1^n \quad \text{and} \quad f_2: V_2 \rightarrow X_2^n \]

where the function \( f_1 \) consists of three functions \( f_{11}, f_{12}, \) and \( f_{13} \) defined as below:

\[ f_{11}: V_{11} \mapsto U_1^n, f_{12}: V_{12} \mapsto W_1^n \quad \text{and} \quad f_{13}: U_1^n \mapsto X_1^n. \]

Similarly, the function \( f_2 \) consists of three functions \( f_{21}, f_{22}, \) and \( f_{23} \) defined as below:

\[ f_{21}: V_{21} \mapsto W_2^n, f_{22}: V_{22} \mapsto U_2^n \quad \text{and} \quad f_{23}: U_2^n \mapsto X_2^n. \]

Let \( P^* \) be the set of probability distributions \( P^*(q, w_1, u_1, w_2, u_2, x_1, x_2) \) that factor as

\[ p_{x_j}(q)p_{w_1}(w_1 | q)p_{w_2}(w_2 | q)p_{u_1}(u_1 | q)p_{u_2}(u_2 | q)p_{x_1}(x_1 | w_1, u_1, q)p_{x_2}(x_2 | w_2, u_2, q), \]

where \( p_{x_j}(q) \) equals 1 or 0 for \( j = 1, 2 \).

Fix \( P^* \). For receiver 1, the set of nonnegative quadruples \((S_1, T_1, S_2, T_2)\) denoted by \( R_m^{(n)}(P^*) \) that satisfy

\[ S_1 \leq I(U_1; Y_1 | W W Q). \tag{2} \]
\[ T_1 \leq I(W_1; Y_1 | W Q). \tag{3} \]
\[ T_2 \leq I(W_2; Y_1 | W Q). \tag{4} \]
\[ S_1 + T_1 \leq I(U_1 W_1; Y_1 | W Q). \tag{5} \]
\[ S_1 + T_2 \leq I(U_1 W_2; Y_1 | W Q). \tag{6} \]
\[ T_1 + T_2 \leq I(W_1;W_1/\mathcal{Y})+(7) \]
\[ S_1 + T_1 + T_2 \leq I(U_1;W_1/\mathcal{Y})+(8) \]
\[ S_1^3 = 0,(9) \]
\[ T_1^3 = 0, (10) \]
\[ T_2^3 = 0. \]

Similarly, for receiver 2, the set of nonnegative quadruples \((S_1, T_1, S_2, T_2)\) denoted by \(R^{(\omega_2)}_{\text{in}}(P')\) that satisfy
\[ S_2 \leq I(U_2;W_2/\mathcal{Y})+(12) \]
\[ T_2 \leq I(W_2;U_2/\mathcal{Y})+(13) \]
\[ T_1 \leq I(W_1;U_1/\mathcal{Y})+(14) \]
\[ S_2 + T_2 \leq I(U_1;W_2/\mathcal{Y})+(15) \]
\[ S_2 + T_1 \leq I(U_2;W_2/\mathcal{Y})+(16) \]
\[ T_1 + T_2 \leq I(W_1;U_2/\mathcal{Y})+(17) \]
\[ S_2 + T_1 + T_2 \leq I(U_2;W_2/\mathcal{Y})+(18) \]
\[ S_2^3 = 0, (19) \]
\[ T_1^3 = 0, (20) \]
\[ T_2^3 = 0. \]

For a set \(S\) of quadruples \((S_1, T_1, S_2, T_2)\), let \(\tilde{\Omega}(S)\) be the set of \((R_1, R_2)\) such that \(0 \leq R_1 \leq S_1 + T_1\) and \(0 \leq R_2 \leq S_2 + T_2\) for some \((S_1, T_1, S_2, T_2) \subseteq S\). The following result holds (Han and Kobayashi,1981).

**Theorem 1 (the original HK rate region)**: The set

\[ R^{\omega}_{\text{in}} = \tilde{\Omega}(S) \cup R^{(\omega_1)}_{\text{in}}(P') \cap R^{(\omega_2)}_{\text{in}}(P') \]

is an achievable rate region for the DM-IC.

**Proof**: Simultaneous superposition coding and simultaneous decoding are used (Han and Kobayashi,1981). Further, the HK rate region can be simplified using Fourier-Motzkin elimination to obtain the following result (Chong, Motani, Garg and Gamal, 2008; Kobayashi and Han, 2007).

**Lemma 1 (the simplified HK rate region)**: let \( R^{\omega}_{\text{in}}(P') \) be the set of \((R_1, R_2)\) satisfying

\[ R_1 \leq I(W_1;U_1/\mathcal{Y})+(23) \]
\[ R_1 \leq I(U_1;W_1/\mathcal{Y}) + I(W_1;U_1/\mathcal{Y})+(24) \]
\[ R_2 \leq I(W_1;U_1/\mathcal{Y})+(25) \]
\[ R_2 \leq I(U_2;W_2/\mathcal{Y}) + I(W_2;U_2/\mathcal{Y})+(26) \]
\[ R_1 + R_2 \leq I(U_1;W_2/\mathcal{Y}) + I(U_2;W_2/\mathcal{Y})+(27) \]
\[ R_1 + R_2 \leq I(U_1;W_2/\mathcal{Y}) + I(U_2;W_2/\mathcal{Y})+(28) \]
\[ R_1 + R_2 \leq I(U_2;W_2/\mathcal{Y}) + I(U_2;W_2/\mathcal{Y})+(29) \]
\[ 2R_1 + R_2 \leq I(U_1;W_1/\mathcal{Y}) + I(U_1;W_1/\mathcal{Y}) + I(W_1;W_2/\mathcal{Y})+(30) \]
\[ R_1 + 2R_2 \leq I(U_1;W_1/\mathcal{Y}) + I(U_1;W_1/\mathcal{Y}) + I(W_1;W_2/\mathcal{Y})+(31) \]
\[ R_1^3 = 0,(32) \]
\[ R_2^3 = 0, (33) \]

where \(Q\) is an auxiliary time-sharing random variable and \(W_j\) and \(U_j\) are the auxiliary random variable denoting the common messages and the private messages of the user \(j, j = 1, 2\). Then the set
is an achievable rate region for the DM-IC.

Now, instead of using simultaneous decoding, we adopt successive cancelation decoding to achieve the HK rate region.

**Theorem 2:** The simplified HK rate region described in Lemma 1 can be achieved using simultaneous superposition coding and successive cancelation decoding.

Proof: Fix \( p \) \( (q)p_{w,\beta}(w_1 | q)p_{u,\beta}(u_1 | q)p_{w,\beta}(w_2 | q)p_{x,\beta}(x_1 | w_1,q)p_{x,\beta}(x_2 | w_2,q) \).

Random codebook construction: Randomly generate a sequence \( q' \), generating each element i.i.d. according to \( \hat{O}^n \). For the codeword \( q' \) and \( j\), \( j = 1,2 \), generate \( 2^{nR_j} \) conditionally independent codewords \( w_1^n(m_{j0}) \), \( m_{j0} \rightarrow M_j = \{1,2,\ldots,2^{nR_j}\} \), generating each element i.i.d. according to \( \hat{O}^n \). For the codeword \( q' \), generate \( 2^{nR} \) conditionally independent codeword \( u_1^n(m_{j1}) \), \( m_{j1} \rightarrow M_j = \{1,2,\ldots,2^{nR_j}\} \), generating each element i.i.d. according to \( \hat{O}^n \).

**Encoding:** Let \( M_{j0} \), \( M_{j1} \), \( M_{20} \), and \( M_{22} \) be four messages sets such that \( M_{j0} = M_{10}, M_{j1} = M_{11}, M_{20} = M_{20}, M_{22} = M_{22} \).

Define the encoding functions \( j_1 : M_{j0}, M_{j1} \rightarrow X^n_1 \), \( j_2 : M_{20}, M_{22} \rightarrow X^n_2 \) by \( (m_{j0},m_{j1}) = f_1(w_{j0},u_{j1} | q) \).

For sender \( j \), \( j = 1,2 \), to send the codeword pair \( m_j = (m_{j0},m_{j1}) \), it sends the corresponding codeword \( X^n(m_{j0},m_{j1}). \)

**Decoding:** At each receiver a virtual-three-user MAC is formed because there are three messages to be decoded. Decoding is performed in three steps via successive cancelation decoding and we will have six decoding orders. Each decoding order determines a rate region (named sub-region) described by ten inequalities using coded time-sharing random variable. The union of these six sub-regions constitutes the capacity region of the virtual-three-user MAC. Without loss of generality, we only provide the decoding proof of one decoding order \( (m_{j0},m_{j1},m_{22}) \) that decoder 1.

Decoder 1 declares that \( \hat{m}_{20} \) is sent if it is the unique message such that \( (q^n,w^n_2(\hat{m}_{20}),y^n_1) \in T^{(e)} \); otherwise it declares an error. If such \( \hat{m}_{20} \) is found, then the decoder 1 looks for the unique message \( \hat{m}_{10} \) such that \( (q^n,w^n_1(\hat{m}_{10}),w^n_2(\hat{m}_{20}),y^n_1) \in T^{(e)} \); otherwise it claims an error. If such \( \hat{m}_{20} \) and \( \hat{m}_{10} \) are found, the decoder 1 determines the unique message \( \hat{m}_{11} \) such that \( (q^n,w^n_1(\hat{m}_{10}),x^n_1(\hat{m}_{10},\hat{m}_{11}),y^n_1) \in T^{(e)} \); otherwise it declares an error.

**Analysis of the probability of error:** We bound the probability of error averaged over codebooks and messages. By symmetry of code generation, the probability of error does not depend on which codeword was sent. Without loss of generality, we assume that \( M_1 = (1,1) \) and \( M_2 = (1,1) \) were sent. Then we have the conditional probability of the error events given the above codeword was sent: \( P(e) = P(e | M_1 = (1,1), M_2 = (1,1)) \).

And define the error events as follows

\[ e_1 = \left\{ (Q^n,W^n_1(1),W^n_2(1),Y^n_1) \notin T^{(e)} \right\}, \]
\[ e_2 = \left\{ (Q^n,W^n_2(\hat{m}_{20}),Y^n_1) \in T^{(e)} \text{ for some } \hat{m}_{20} \neq 1 \right\}, \]
\[ e_3 = \left\{ (Q^n,W^n_1(\hat{m}_{10}),W^n_2(1),Y^n_1) \in T^{(e)} \text{ for some } \hat{m}_{10} \neq 1 \right\}, \]
\[ e_4 = \left\{ (Q^n,W^n_1(1),W^n_2(\hat{m}_{11}),Y^n_1) \in T^{(e)} \text{ for some } \hat{m}_{11} \neq 1 \right\}. \]
Then, by the union bound of events, we have $P(\varepsilon) \leq P(\varepsilon_1) + P(\varepsilon_2) + P(\varepsilon_3) + P(\varepsilon_4)$. By the law of large numbers, $P(\varepsilon_i) \to 0$ as $n \to \infty$. Since for $m_{20} \to 1$, $W_2^*(m_{20})$ is conditionally independent of $(W_2^*(1), Y_2')$ given $Q$. By the packing lemma, $P(\varepsilon_i) \to 0$ as $n \to \infty$ if $R_{20} < I(W_2; Y_1, Q) - \delta(\tilde{o})$ and $P(\varepsilon_i) \to 0$ as $n \to \infty$ if $R_{11} < I(U_1; Y_2, W_2; Q) - \delta(\tilde{o}) = I(U_1; Y_2, W_2; Q) - \delta(\tilde{o})$. Thus the total average probability of decoding error $P(\varepsilon) \to 0$ as $n \to \infty$. Further, considering all the sum rates of these three individual rates, we will have the rate sub-region for this decoding order given below.

$$R_{11} \leq I(U_1; Y_1 | W_1, W_2, Q),$$

$$R_{10} \leq I(W_1; Y_1 | W_2, Q),$$

$$R_{20} \leq I(W_2; Y_1 | W_1, Q).$$

$$R_{11} + R_{10} \leq I(U_1; Y_1 | W_1, W_2, Q) + I(W_1; Y_1 | W_2, Q) = I(U_1; Y_1 | W_2, Q),$$

$$R_{10} + R_{20} \leq I(W_1; Y_1 | W_2, Q) + I(W_2; Y_1 | W_1, Q),$$

$$R_{11} + R_{20} \leq I(U_1; Y_1 | W_1, W_2, Q) + I(W_1; Y_1 | W_2, Q) = I(U_1 W_1 W_2; Y_1 | Q).$$

Similarly at receiver 2, we have

$$R_{22} \leq I(U_2; Y_2 | W_2, Q),$$

$$R_{20} \leq I(W_2; Y_2 | W_1, Q),$$

$$R_{20} \leq I(W_2; Y_2 | W_1, Q),$$

$$R_{20} + R_{22} \leq I(U_2 W_2; Y_2 | W_1, Q),$$

$$R_{20} + R_{10} \leq I(U_2 W_1 W_2; Y_2 | Q).$$

$$R_{20} \leq 0.$$
By using the Fourier-Motzkin elimination technique on these twenty inequalities, we have the conclusion of Theorem 2.

4.2. To Achieve the CMG Rate Region

In 2006, a rate region named the CMG rate region is achieved by using sequential superposition coding in conjunction with simultaneous decoding. In the sequential superposition coding, three auxiliary random variables \( Q, W_1, \) and \( W_2 \) defined on arbitrary finite sets \( Q, W_1, \) and \( W_2 \), respectively. The auxiliary random variables \( W_1 \) and \( W_2 \) server as cloud centers that can be distinguished by both receivers. For sender \( i, \) \( i = 1, 2, \) instead of generating two independent codebooks with codewords \( w_i(j) \) and \( u_i(k) \) like simultaneous superposition coding, for each codeword \( w_i(j) \), it generates a codebook with codewords \( n_{ij}(x) \). The CMG rate region described by the three auxiliary random variables is given below (Chong, Motani, Garg and Gamal, 2008).

Lemma 2 (the original CMG rate region): Let \( P_1^* \) be the set of probability distributions \( P_1^*(q,w_i,w_j,x_i,x_j) \) that factor as

\[
P_1^*(q,w_i,w_j,x_i,x_j) = P_{q|w_i}(w_i | q)p_{w_i|q}(w_i | q)p_{x_i|w_i|q}(x_i | w_i)p_{x_j|w_j}(x_j | w_j).
\]

For a fixed \( P_1^* \), let \( R_{CMG}^{(1)}(P_1^*) \) be the set of nonnegative quadruples \( (S_1, T_1, S_2, T_2) \) satisfying

\[
S_1 \leq I(X_1;Y_1 | W_1W_2Q),
\]

\[
S_1 + T_1 \leq I(X_1;Y_1 | W_1Q),
\]

\[
S_1 + T_2 \leq I(X_1W_2;Y_1 | W_2Q),
\]

\[
S_1 + T_1 + T_2 \leq I(X_1W_2;Y_1 | Q),
\]

\[
S_1^3 \leq 0,
\]

\[
T_1^3 \leq 0,
\]

\[
T_2^3 \leq 0.
\]

Similarly, for receiver 2, the set of nonnegative quadruples \( (S_1, T_1, S_2, T_2) \) denoted by \( R_{CMG}^{(2)}(P_2^*) \) that satisfy

\[
S_2 \leq I(X_2;Y_2 | W_1W_2Q),
\]

\[
S_2 + T_2 \leq I(X_2;Y_2 | W_2Q),
\]

\[
S_2 + T_1 \leq I(X_2W_1;Y_2 | W_1Q),
\]

\[
S_2 + T_1 + T_2 \leq I(X_2W_1;Y_2 | Q),
\]

\[
S_2^3 \leq 0,
\]

\[
T_1^3 \leq 0,
\]

\[
T_2^3 \leq 0.
\]

Then the set

\[
R_{CMG} = \bigcup_{P_1^* \in P_1^*} \bigcup_{P_2^* \in P_2^*} R_{CMG}^{(1)}(P_1^*) \cap R_{CMG}^{(2)}(P_2^*)
\]

is an achievable rate region for the DM-IC.

Further, the original CMG rate region can be simplified using Fourier-Motzkin elimination to obtain the following result (Chong, Motani, Garg and Gamal, 2008).

Lemma 3 (the simplified CMG rate region): For a fixed \( P_1^* \), let \( R_{CMG}^{'}(P_1^*) \) be the set of \((R_x, R_y)\) satisfying
Then the following set \( R^* = \bigcup_{P_j^+} R_{CMG}^{CMG}(P_j^+) \) is an achievable rate region for the DM-IC.

After removing the redundant inequalities (80) and (82) the following result holds (Chong, Motani, Garg and Gamal, 2008).

**Theorem 3 (the compact CMG rate region):** For a fixed \( P_j^+ \) \( \hat{=} \) \( P_j^+ \), let \( R_{CMG}^{CMG}(P_j^+) \) be the set of \((R_1, R_2)\) satisfying

\[
R_1 \leq I(X_1; Y_1 \mid W_1 Q), \\
R_2 \leq I(X_2; Y_2 \mid W_2 Q), \\
R_1 + R_2 \leq I(X_1; Y_1 \mid W_1 Q) + I(X_2; Y_2 \mid W_1 Q), \\
R_1 + R_2 \leq I(X_1; Y_1 \mid W_2 Q) + I(X_2; Y_2 \mid W_2 Q), \\
R_1 + R_2 \leq I(X_1; Y_1 \mid W_2 Q) + I(X_2; Y_2 \mid W_1 Q), \\
2R_1 + R_2 \leq I(X_1; Y_1 \mid W_2 Q) + I(X_1; Y_1 \mid W_2 Q) + I(X_2; Y_2 \mid W_2 Q), \\
R_1 + 2R_2 \leq I(X_1; Y_1 \mid W_2 Q) + I(X_1; Y_1 \mid W_2 Q) + I(X_2; Y_2 \mid W_2 Q),
\]

Then the following set \( R^* = \bigcup_{P_j^+} R_{CMG}^{CMG}(P_j^+) \) is an achievable rate region for the DM-IC.

**Theorem 4:** The simplified CMG rate region described in Lemma 3 can be achieved using sequential superposition coding and successive cancelation decoding.

**Proof:** Since the proof is similar to the proof of Theorem 2 we only give a brief one as follows. Due to the use of sequential superposition coding, we have to decode the common messages of each user before decoding the private messages. So we have three decoding orders at each receiver. Since the receiver is not concerned with the decoding of the other users’ common messages after its own messages have been decoded correctly, So the associated decoding order is unnecessary, which implies that the rate expressions at each receiver can be reduced (Chong, Motani, Garg and Gamal, 2008; Kobayashi and Han, 2007). By removing the redundant inequalities (45), (46), (48), (55), (56) and (58) first, then by using the Fourier-Motzkin algorithm, we can have the result.
4.3. On the Equivalence between the Two Regions

One can readily see that $R^*_HK = R^*_HK$ and $R^*_{CMG} = R^*_{CMG}$ since $R^*_HK$ and $R^*_{CMG}$ are obtained by using Fourier-Motzkin elimination on $R^*_{HK}$ and $R^*_{CMG}$, respectively. The key question is how to prove that $R^*_{HK} = R^*_{CMG} = R^*_{CMG}$? This has been solved by showing that $R^*_{CMG} = R^*_{HK}$ first, then $R^*_{HK} = R^*_{CMG}$ (Chong, Motani, Garg and Gamal, 2008). Their methods analyze a possible value of probability distribution when the two rate regions are equivalent, which is intangible.

In this subsection, we will illustrate that $R^*_{HK} = R^*_{CMG}$ when the two rate regions have the same probability distribution. Now, we analyze all the decoding orders when simultaneous superposition coding or sequential superposition coding together with successive cancelation decoding are used to obtain the rate region of the IC. At each receiver, a virtual-three-user MAC forms because there are three messages to be decoded. Since simultaneous superposition coding doesn’t have constraints on successive cancelation decoding, we will have six decoding orders to achieve the capacity region of the virtual-three-user MAC when simultaneous superposition coding and successive cancelation decoding are used. These six decoding orders are $\hat{m}_{20}, m_{10}, m_{11}$, $\hat{m}_{10}, m_{20}, m_{11}$, $\hat{m}_{20}, m_{11}, m_{10}$, $\hat{m}_{11}, m_{20}, m_{10}$, $\hat{m}_{11}, m_{10}, m_{20}$, and $\hat{m}_{10}, m_{11}, m_{20}$ for receiver 1. The capacity region of the virtual-three-user MAC is determined by three lower bounds and three upper bounds on individual rates, three upper bounds on the sum rate of any two messages and one upper bound on the sum rate of three messages. In fact, the upper bounds of the receiver’s common message, the other receiver’s common message, and the sum of the two messages are redundant from the view of mathematics. So four upper bounds are left, i.e., the upper bounds on the receiver’s private message, the sum of the receiver’s private message and the receiver’s common message, the sum of the receiver’s private message and the other receiver’s common message, and the sum of the receiver’s private message, the receiver’s common message, and the other receiver’s common message. However, each decoding order only can determine a rate sub-region including one maximal individual upper bound, one maximal upper bound on the sum rate of any two messages and one maximal upper bound on the sum rate of three messages. Hence, to obtain the rate region satisfying the above conditions, we only need to take the union of rate sub-regions obtained by two instead of six decoding orders. They are $\hat{m}_{20}, m_{10}, m_{11}$ and $\hat{m}_{10}, m_{20}, m_{11}$ for receiver 1. From the Angle of decoding, this can be explained as follows. Firstly, the intended private messages are decoded early or late relative to the intended common messages does not affect user’s individual and sum rates because of the symmetry. So we can fix the intended private messages to be decoded later than the intended common messages, which is just sequential superposition coding. That means that three of the decoding orders are unnecessary, e.g., $\hat{m}_{20}, m_{10}, m_{11}$, $\hat{m}_{10}, m_{20}, m_{11}$, and $\hat{m}_{10}, m_{20}, m_{11}$ are necessary for receiver 1. Secondly, the receiver is not concerned with the decoding of the other user’s common messages after its own messages have been decoded correctly. So the decoding order in which the rival’s common messages are decoded last is unnecessary, e.g., $\hat{m}_{20}, m_{10}, m_{11}$ and $\hat{m}_{10}, m_{20}, m_{11}$ are necessary for receiver 1. Thus, at each receiver, only two decoding orders in which the intended private messages decoded last are necessary. In fact, they are just the same as those of sequential superposition coding, which provides some insight into the equivalence between the two rate regions.

5. CONCLUSIONS

In this paper, we have shown that successive cancelation decoding together with the same superposition coding can also achieve the HK and the CMG rate regions, respectively, which provides some insight into the relationship between the two forms of superposition coding from the perspective of decoding. The discussions on the decoding orders of the messages in detail are helpful to understand the two rate regions, and can also intuitively explain the equivalence between the two rate regions from a new perspective.

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