Coefficient Numerical Inversion of Temperature Model with Hot-Pressing of Fiberboard

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Abstract

This paper presents a multigrid algorithm for the solution of coefficient inverse problems constrained by variable-coefficient linear reaction-diffusion partial differential equations. The aim of this work is to assess thermal conductivity of fiberboards at two different kinds of the raw materials (miscellaneous wood composite materials and log composite materials). Thermal measurements are done conducted with Neumann boundary conditions. Inversion variable is considered to be a function of space and temporal and the relationship between space and time is found to a second order polynomial function. The inverse problem is formulated as a PDE-constrained optimization. The discrete control equation by using central difference approach in this study, thentake into account Newton iterative equation for inversion parameter and full observations is used to analyze the performance of the scheme for the non-constant coefficient case. Numerical experiments demonstrate that the effectiveness of the method for different materials. Finally the results shows that calculated results agreed comparatively well with the experiment measured value in two materials of hot-pressing, which proved that the analytical result is useful to calculate the thermal conductivity of fiberboard during hot-pressing.

Key words: Hot-Pressing, Thermal Conductivity, Inverse Problem, Numerical Solution

1. INTRODUCTION

In the production of fiberboard, the hot pressing was a complex process involving several heat and mass transfer properties of the fiber mat. A sensitivity study of the heat and mass transfer model to mat physical and mechanical properties has performed in some study (Humphrey and Bolton, 1989; Carvalho and Costa, 1998; Carvalho and Costa, 2003). Thermal conductivity is important parameter affecting heat transfer of fiberboards. For example, see the work OfUkrainczyk (2009) for a general discussion. The phenomenon of heat conduction constrained by Fourier law, the basic equation is as follows:

\[ q = -\frac{a \Delta T}{H} \]

where \( H \) corresponds to the thickness of tablet and \( \Delta T \) is defined as plate material surface temperature difference of up and down, \( a \) is the thermal conductivity, \( q \) is the heat flux through the tablet.

It can be seen that the thermal conductivity \( a \) is a physical quantity which contact temperature gradient of heat transfer with heat flow density vector. It is determined by that point of the intrinsic material. On the numerical value, the thermal conductivity is equal to the temperature gradient in the unit under the action of objects within unit time through unit area of heat, and the main indicators reflect the material heat conduction ability (Stanislaw Chudzik 2009).

In this study, a multigrid algorithm is presented for the solution of coefficient inverse problems and the one-dimensional mathematical model of the hot-pressing is considered, this because mathematical models were recognized as important tools for optimization, control and scheduling. For an excellent review a promising work for problem similar to ours can be found in (Dai and Yu, 2004) and (Arun Gupta, 2013). However all these models take into account thermal conductivity as a fixed parameter. Such problems were seen in (Norberto and Mario 2001). In this paper, there exists significant work on thermal conductivity for an unknown function problem which was discussed in numerical solution of hot-pressing of fiberboard. For example, see the work of (Sonderegger and Niemz 2009; Liu Wang and Cao 2014; Hussein and Lesnic 2015) for a general discussion.
The mathematical model was based on the work of (Kavazovic, Deteix, Cloutier, Fortin 2010). The model is as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= a \frac{\partial^2 u}{\partial x^2} \quad 0 < x < l, 0 < t < T \quad (1) \\
\frac{\partial u}{\partial x} \bigg|_{x=0} &= 0 \quad 0 < x < l, 0 < t < T \quad (2) \\
\frac{\partial u}{\partial x} \bigg|_{x=l} &= T_p \quad 0 < x < l, 0 < t < T \quad (3) \\
u(x,0) &= T_0 \quad 0 < x < l, 0 < t < T \quad (4)
\end{align*}
\]

where \( a = a(x,t) \) is thermal conductivity of fiberboards.

The organization of this paper is as follows. Section 2.1 presents a detail discrete process for the equation (1)-(4). Section 2.2 presents a detail iterative process. In particular, recast as nonlinear least-squares minimization problems with the unknown thermal conductivity. Section 2.3 presents several experimental results for hot-pressing of fiberboard. Section 3 concludes this paper with a summary of the proposed algorithm and presents main contribution.

2. THERMAL CONDUCTIVITY INVERSE PROBLEM IN HOT-PRESSING

2.1 Model of Discrete

The scope of inverse problems has existed in various branches of physics, engineering and mathematics for a long time. The theory of inverse problems has been extensively developed within the past decade due partly to its importance in applications; on the other hand the numerical solutions to such problems need huge computations and also reliable numerical methods. For a stable stationary solution of thermal conductivity, a multigrid algorithm for the solution of coefficient inverse problems is somewhat similar to (Santi S and George2008; Wu, Wang and Zhang2015).

More precisely, the following describes discretization of a reaction-diffusion equation (1) and a numerical computation scheme in one-dimensional space. Space variables \( x \) and a time variable \( t \) are discretized with finite differences: \( ih \) in space and \((j + \frac{1}{2})\tau) \) in time, as follows:

\[
x_i = ih \quad i = 0,1,2,\ldots,m \quad t_j = (j + \frac{1}{2})\tau \quad j = 0,1,2,\ldots,n
\]

where

\[
h = \frac{l}{m}, \quad \tau = \frac{T}{n}
\]

supplementary definition:

\[
x_{-1} = -h, \quad x_{m+1} = l + h
\]

then, the relation between the variable \( u(x,t) \) and its discrete expression \( u_{ij} \) becomes \( u_{ij} = u(x_i,t_j) \).

The next equations \( \Delta_t u \) and \( \Delta_x u \) respectively describe the discretised versions of \( \frac{\partial u}{\partial t} \) and \( \frac{\partial u}{\partial x} \) as follows: then for model (1) at the node \((x_i,t_j)\) take central difference discrete

\[
\Delta_t u = \frac{\partial u(x_i,t_{j+\frac{1}{2}})}{\partial t} \approx \frac{u(x_i,t_{j+1}) - u(x_i,t_j)}{\tau} \quad (5)
\]

\[
\Delta_x u = \frac{\partial u(x_{i+\frac{1}{2}},t_j)}{\partial x} \quad \approx \frac{u(x_{i+1},t_j) - u(x_i,t_j)}{h} \quad (6)
\]

\[
\Delta_t u = \frac{\partial u(x_{i-\frac{1}{2}},t_{j+\frac{1}{2}})}{\partial t} \quad \approx \frac{u(x_{i-1},t_{j+1}) - u(x_i,t_{j+\frac{1}{2}})}{\tau} \quad (7)
\]

\[
\Delta_x u = \frac{\partial u(x_{i+\frac{1}{2}},t_{j+\frac{1}{2}})}{\partial x} \quad \approx \frac{u(x_{i+1},t_{j+\frac{1}{2}}) - u(x_i,t_{j+\frac{1}{2}})}{h} \quad (8)
\]

with Eq. (5)-(8) the reaction-diffusion equation of Eq. (1) becomes a discretised version of a linear equation:
in which $i = 0, 1, 2, \ldots, m$ and $j = 0, 1, \ldots, n$.

Take the following form of approximate:

$$u_{i,j} \approx \frac{u_{i,j-1} + u_{i,j+1}}{2} \quad \text{and} \quad u_{i,j} \approx \frac{u_{i,j-1} + u_{i,j+1}}{2} \quad i = 0, \ldots, m-1; \quad j = 0, \ldots, n-1$$

then equation (9) can be written as

$$\frac{a(x_t,t)}{2h^2} u_{i+1,j} - u_{i,j} + \frac{1}{\tau} u_{i,j+1} + \frac{a(x_t,t)}{2h^2} u_{i+1,j+1} = -\frac{a(x_t,t)}{h} \quad \text{(10)}$$

Discrete boundary conditions at $x = 0$ and $x = l$:

$$u_{0,j} - u_{-1,j} = 0, \quad \frac{u_{m,j} - u_{m-1,j}}{2h} = T_p$$

Then the variable $u$ was expressed as follows

$$u_{i,j} = u_{m+1,0} - 2hT_p \quad \text{(11)}$$

When $j = 0, t_j = 0, u(x,0) = T_0$, the reaction-diffusion equation (10) becomes a discretized equation:

$$\frac{a(x_t,t)}{2h^2} u_{i+1,j} - u_{i,j} + \frac{1}{\tau} u_{i,j+1} + \frac{a(x_t,t)}{2h^2} u_{i+1,j+1} = -\frac{a(x_t,t)}{h} \quad \text{(12)}$$

When $i = 0$ the equation (10) becomes the following equation:

$$\frac{a(x_t,t)}{h^2} u_{i+1,j} - \frac{1}{\tau} u_{i,j+1} + \frac{a(x_t,t)}{h^2} u_{i+1,j+1} = -\frac{a(x_t,t)T_p}{h} \quad \text{(13)}$$

When $i = m$ the equation (10) can be rewritten as:

$$\frac{a(x_t,t)}{h^2} u_{m-1,j} - \frac{1}{\tau} u_{m,j+1} + \frac{a(x_t,t)}{h^2} u_{m,j+1} = -\frac{a(x_t,t)T_p}{h} \quad \text{(14)}$$

When $1 \leq i \leq m - 1$ the equation (10) as shown in the following:

$$\frac{a(x_t,t)}{2h^2} u_{i+1,j} - \frac{1}{\tau} u_{i,j+1} + \frac{a(x_t,t)}{2h^2} u_{i+1,j+1} = -\frac{a(x_t,t)}{h} \quad \text{(15)}$$

Compile matrix equation:

$$QU_i = V_i$$

where $Q$ is $(m+1) \times (m+1)$ three strictly diagonally dominant matrix and

$$U_i = [u_{0,1}, u_{0,2}, \ldots, u_{0,m}]^T \quad V_i = [-\frac{T_0}{\tau}, -\frac{T_0}{\tau}, \ldots, -\frac{T_0}{\tau} - \frac{aT_p}{h\tau}]$$

When $1 \leq j \leq n - 1$:

When $i = 0$, equation (10) can be written as:

$$\frac{a(x_t,t)}{2h^2} u_{1,j} - u_{0,j+1} + \frac{1}{\tau} u_{0,j+1} + \frac{a(x_t,t)}{h^2} u_{1,j+1} = \frac{a(x_t,t)}{h^2} u_{1,j+1} - \frac{1}{\tau} u_{0,j} - \frac{a(x_t,t)}{h} u_{0,j} \quad \text{(16)}$$

When $i = m$, equation (10) can be written as:

$$\frac{a(x_t,t)}{2h^2} u_{m-1,j} - \frac{1}{\tau} u_{m,j+1} + \frac{a(x_t,t)}{h^2} u_{m,j+1} = \frac{a(x_t,t)}{h^2} u_{m,j+1} - \frac{1}{\tau} u_{m,j} + \frac{a(x_t,t)}{h} u_{m,j} \quad \text{(17)}$$

When $1 \leq i \leq m - 1$, is the equation (10) itself. Compile matrix equation:

$$Q_jU_{j+1} = W_jU_j + V_{j+1}$$
where

\[ U_j = [u_{0,j}, u_{1,j}, u_{2,j}, \ldots, u_{m,j}]^T \quad V_{j+1} = [0, 0, \ldots, 0, -\frac{2aT}{h}]^T \]

\( Q_{j+1} \) can be written as

\[ Q_{j+1} = p_{j+1} - \frac{1}{\tau} E \]

and \( W_{j+1} \) can be written as

\[ W_{j+1} = -p_{j+1} - \frac{1}{\tau} E \]

the main diagonal elements of \( p_{j+1} \) is \(-\frac{a}{h^2}\), \( (j = 0, 1, \ldots, m) \)

upper triangular diagonal elements:

\[ \left( \frac{a}{h^2}, \frac{a}{2h^2}, \ldots, \frac{a}{2h^2} \right), \quad (i = 0, 1, \ldots, m-1) \]

lower triangular diagonal elements

\[ \left( \frac{a}{2h^2}, \ldots, \frac{a}{2h^2}, \frac{a}{h^2} \right), \quad (i = 1, \ldots, m) \]

For given initial condition and boundary of Eq. (2)-(4), thermal conductivity was computed by solving linear equation described by Eq.(10). Thus this can be obtained a time-evolving solution. The multigrid algorithm (Santi, George 2008), for example, provides a solution for linear equation.

2.2 The Iterative Format of Thermal Conductivity

Much attention has been focused on thermal conductivity as a fixed parameter due to it commonly was obtained by experimentally, and was conductive to numerical compute. However, until now, only a limited number of studies have been addressed on thermal conductivity as a function. As is known to all there were many factors affecting thermal properties, for example temperature, moisture, heat capacity, and so on. So in this study an unknown function \( a(x, t) \) was considered.

More precisely, the following error function was selected:

\[ T = \min \sum_{j=0}^{n} |u_{j,j}(b_0, b_1, b_2, b_3, b_4) - u_{j,j}|^2 \]

reconstruct a function of thermal conductivity:

\[ a(x, t) = b_0 + b_1 x + b_2 t + b_3 x^2 + b_4 t^2 \]

using central difference approximation:

\[ g_0 = \frac{T}{\partial b_i} \approx \frac{T(b_0 + dd; b_1, b_2, b_3, b_4) - T(b_0 - dd; b_1, b_2, b_3, b_4)}{2dd} \]

with the second order finite difference approximation:

\[ g_0 = \frac{\partial^2 T}{\partial b_i \partial b_j} \approx \frac{1}{4dd^2} \left[ T(b_0 + dd, b_1 + dd, b_2, b_3, b_4, b_5) - T(b_0 - dd, b_1 + dd, b_2, b_3, b_4, b_5) \right. \]

\[ - T(b_0 + dd, b_1 - dd, b_2, b_3, b_4, b_5) + T(b_0 - dd, b_1 - dd, b_2, b_3, b_4, b_5) + T(b_0 + dd, b_1, b_2, b_3, b_4, b_5) - 2T(b_1, b_2, b_3, b_4, b_5) + T(b_0 - dd, b_1, b_2, b_3, b_4, b_5) \]

by forming a Taylor expansion, Newton iteration formats as follows:

\[ A \Delta b = B \]

where

\[
A = \begin{bmatrix}
    g_{00} & g_{01} & g_{02} & g_{03} & g_{04} & g_{05} \\
    g_{10} & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} \\
    g_{20} & g_{21} & g_{22} & g_{23} & g_{24} & g_{25} \\
    g_{30} & g_{31} & g_{32} & g_{33} & g_{34} & g_{35}
\end{bmatrix},
B = \begin{bmatrix}
    -g_0 \\
    -g_1 \\
    -g_2 \\
    -g_3 \\
    -g_4 \\
    -g_5
\end{bmatrix}, \quad \Delta b = \begin{bmatrix}
    \Delta b_0 \\
    \Delta b_1 \\
    \Delta b_2 \\
    \Delta b_3 \\
    \Delta b_4 \\
    \Delta b_5
\end{bmatrix}
\]
and $\Delta b_0$, $\Delta b_1$, $\Delta b_2$, $\Delta b_4$, $\Delta b_5$ are iterative improvement increment, respectively. $dd$ was the number of discrete step of partial derivative. Regard to $b_0$, $b_1$, $b_2$, $b_4$, $b_5$ the same discrete step $dd$ ($dd = 0.01$).

So far space and time were discrete and Newton iteration formats were found by forming a Taylor expansion. Based on these decompositions, the following consider two cases of thermal conductivity inverse problem with two kinds of hybrid materials.

### 2.3 Further Numerical Experiments

Commercially available, wet-processed insulation fiberboards were provided by Hunan (M.S. Hussein, D. Lesnic. 2015). In this paper two kinds of the raw materials were considered: miscellaneous wood composite materials and log composite materials, respectively.

#### Case one: miscellaneous wood composite materials

First of all take into account the same grid subdivision in time and space $m = n = 10$, $fix h = 1$, $\tau = 0.1$, then choose $t < 100$, $s = \frac{m}{2} + 1$ and temperature measurements:

$$u_e = [42, 58, 120, 140, 150, 156, 158, 160, 162, 165]$$

after five times iteration the result of $T = 756.4827$

at the same time the result of $b$ is as follows:

$$b = [1.6070, 1.7147, -3.3072, -2.2592, 0.3167, 1.6516]$$

so the approximate expression of coefficient $a(x, t)$ can be expressed:

$$a(x, t) = 1.6070 + 1.7147x - 3.3072t - 2.2592xt + 0.3167x^2 + 1.6516t^2$$

![Figure 1](image1.png)

**Figure 1.** The relationship between temperature measurement and coefficient inversion calculated value of miscellaneous wood mixed material.

#### Case two: log composite materials

Choose the same grid subdivision in time and space and initial value as the case one, but take the different temperature measurements

$$u_e = [26, 27, 37, 80, 105, 110, 112, 114, 116, 120]$$

then after eight times iteration the result of $T = 836.7633$

at the same time the result of $b$ is as follows:

$$b = [1.2946, 1.6434, -2.2232, -2.2697, 0.3800, 0.9613]$$

so the approximate expression of coefficient $a(x, t)$ can be expressed:

$$a(x, t) = 1.2946 + 1.6434x - 2.2232t - 2.2697xt + 0.3800x^2 + 0.9613t^2$$
show the relationship between temperature measurement and coefficient inversion calculated value of undersized log composite materials.

3. CONCLUSIONS

In this paper, a major motivation is to presents a multigrid algorithm for the inverse problems with linear reaction-diffusion partial differential equations constraints. Our numerical experiments gave promising results. Figures 1 and 2 show the relationship between temperature measurement and coefficient inversion calculated value of miscellaneous wood mixed material. Figure 2 show the relationship between temperature measurement and coefficient inversion calculated value of undersized log composite materials. Through analysis of the experimental data and the simulative result, Figures 1 and 2 shown that the thermal conductivity of the heat transfer process is reasonable, and it would be used to further simplify the hot-pressing process of fiberboard.

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